Optimal control problems with horizon tending to infinity and lacking controllability assumption

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M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems,(*submitted*) (2009)

An Optimal Control Problem

$$V_t(y_0) := \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds,$$

where $s \mapsto y(s,u,y_0)$ **denotes the solution to**

$$y'(s) = g(y(s),u(s)), \quad y(0) = y_0.$$

$g : I\!R^d \times U \to I\!R^d$ **Lipschitz,** U **compact,** g h **bounded.**

PROBLEM : Existence of a limit of $V_t(y_0)$ **as** $t \to +\infty$**.**

No ergodicity condition here (Lions-Papanicolaou- Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, Fathi...)The limit may depend on the initial condition

Contents

Introduction

Definition 1 ***The problem*** $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$ ***has a*** *limit value* ***if***

$$V(y_0) := \lim_{t\to\infty} V_t(y_0) = \lim_{t\to\infty} \inf_{u\in\mathcal{U}} \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds.$$

Definition 2 ***The problem*** $\Gamma(y_0)$ ***has a*** *uniform value* ***if it has a limit value*** $V(y_0)$ ***and if:***

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t}\int_{s=0}^{t} h(y(s,u,y_0),u(s))ds \leq V(y_0) + \varepsilon.$$

Examples

• **Example 1: here** $y \in IR^2$ **(seen as the complex plane** $i^2 = -1$**), there is no control**

$$y'(t) = i\, y(t),$$

$$V_t(y_0) \xrightarrow[t \to \infty]{} \frac{1}{2\pi |y_0|} \int_{|z|=|y_0|} h(z) dz,$$

and since there is no control, the value is uniform.

• **Example 2: in the complex plane again, but now** $g(y,u) = i\; y\; u$**, where** $u \in U$ **a given bounded subset of** IR**, and** h **is continuous in** y**.**

Assumptions and Notations

$$\begin{cases} \textbf{The function } h : I\!R^d \times U \longrightarrow I\!R \textbf{ is measurable and bounded} \\ \exists L \geq 0, \forall (y, y') \in I\!R^{2d}, \forall u \in U, \ \|g(y,u) - g(y',u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y,u) \in I\!R^d \times U, \ \|g(y,u)\| \leq a(1 + \|y\|) \end{cases}$$

(HK) $\exists$ **a compact invariant set** K **for the control system**

Average cost induced by u **between** 0 **and** t **by:**

$$\gamma_t(y_0, u) := \frac{1}{t}\int_0^t h(y(s,u,y_0), u(s))ds, \ V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u).$$

$$\textbf{for } m \geq 0\textbf{,} \ \gamma_{m,t}(y_0, u) := \frac{1}{t}\int_m^{m+t} h(y(s,u,y_0), u(s))ds,$$

A Classical Controlability Approach

Suppose that $\exists T > 0, \forall (y_1, y_2) \in K, \exists t \leq T, \forall u \in \mathcal{U}, \ \exists v \in \mathcal{U}, \|y(t,u,y_1) - y(t,v,y_2)\| = 0.$

Then for any $t \geq T$ **and any** $\Psi \in C(K)$ **the maps**

$$y_0 \mapsto V_t^{\Psi}(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{t} h(y(s,u,y_0))ds + \Psi(y(t,u,y_0)),$$

are equicontinuous with a modulus of continuity which does not depend on t **and** Ψ **(but only on the Lipschitz constants of** h **and** f**).**

Thus V_t^Ψ is more regular than Ψ. This also could be obtained and generalized using HJB results with coercive concave hamiltonians.

• **Example 3:** $g(y,u) = -y+u$, where $u \in U$ a given bounded subset of $I\!R^d$, and h is continuous in y.

• **Example 4:** in $I\!R^2$. The initial state is $y_0 = (0,0)$ and $U = [0,1]$, and the cost is $h(y) = 1 - y_1(1-y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1-y_1(s)) \\ u^2(s)(1-y_1(s)) \end{pmatrix}.$$

One can easily observe that the reachable set $G(y_0) \subset [0,1]^2$.

If $u = \varepsilon > 0$ constant, $y_1(t) = 1 - \exp(-\varepsilon t)$ and $y_2(t) = \varepsilon y_1(t)$. So we have $V_t(y_0) \xrightarrow[t\to\infty]{} 0$. Existence of a Uniform Value

No ergodicity :

$$\{y \in [0,1]^2, \lim_{t\to\infty} V_t(y) = \lim_{t\to\infty} V_t(y_0)\} = [0,1] \times \{0\},$$

and starting from y_0 it is possible to reach no point in $(0,1] \times \{0\}$.

A first result in Nonexpansive case

Denote by $G(y_0) := \{y(t,u,y_0), t \geq 0, u \in \mathcal{U}\}$ **the reachable set**

Theorem 3 $h(y,u) = h(y)$ ***only depends on the state,***
$G(y_0)$ ***is bounded (invariant),***
$\forall (y_1, y_2) \in G(y_0)^2$***,*** $\sup_{u \in U} \inf_{v \in U} < y_1 - y_2, g(y_1,u) - g(y_2,v) > \leq 0.$

Then $\Gamma(y_0)$ ***has a*** *limit value* $V_t(y_0) \xrightarrow[t \to +\infty]{} V^*(y_0)$***. The convergence of*** $(V_t)_t$ ***to*** V^* ***is uniform over*** $G(y_0)$***, and the value of*** $\Gamma(y_0)$ ***is uniform.***

A Crucial Technical Lemma

We define $V^-(y_0) := \liminf_{t\to+\infty} V_t(y_0)$**,**
$V^+(y_0) := \limsup_{t\to+\infty} V_t(y_0)$**.**

Lemma 4 ***For every*** m_0 ***in*** $I\!R_+$***, we have:***

$$\sup_{t>0} \inf_{m\le m_0} V_{m,t}(y_0) \ge V^+(y_0) \ge V^-(y_0) \ge \sup_{t>0} \inf_{m\ge 0} V_{m,t}(y_0).$$

Definition 5

$$V^*(y_0) = \sup_{t>0} \inf_{m\ge 0} V_{m,t}(y_0).$$

Sketch of the proof of the first result

Lemma 6 $\forall T > 0, \forall \varepsilon > 0, \forall (y_1, y_2) \in G(y_0)^2, \forall u \in \mathcal{U},\ \exists v \in \mathcal{U},$

$$\forall t \in [0, T],\ \|y(t, u, y_1) - y(t, v, y_2)\| \leq \|y_1 - y_2\| + \varepsilon.$$

Proposition 7 $\forall \varepsilon > 0, \exists m_0, \sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0) + 2\varepsilon$

• $(V_T(y_0))_{T>0}$ **is equicontinuous (Lemma 6 +continuity of** h**)**
• **Define** $G^m(y_0) := \{y(t,u,y_0), t \leq m, u \in \mathcal{U}\}$ **the reachable set in time** m**.**

$$\forall \varepsilon, \exists m_0, \forall z \in G(y_0), \exists z' \in G^{m_0}(y_0) \text{ such that } \|z - z'\| \leq \varepsilon.$$

• **We have** $\inf_{m\geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$**, and** $\inf_{m\leq m_0} V_{m,t}(y_0$ $\inf\{V_t(z), z \in G^{m_0}(y_0)\}$**. By steps 1 and 2** $\inf\{V_t(z), z \in G^{m_0}(y_0)\} \leq \inf\{V_t(z), z \in G(y_0)\} + 2\varepsilon$.

Generalizations

Theorem 8 $\exists\ C^1\ \Delta : IR^d \times IR^d \longrightarrow IR_+$, ***vanishing on the diagonal*** ($\Delta(y,y)=0$) ***and symmetric*** ($\Delta(y_1,y_2)=\Delta(y_2,y_1)$)
$h(y,u)=h(y)$ ***only depends on the state,***
$G(y_0)$ ***is bounded (invariant),***
$\forall (y_1,y_2) \in G(y_0)^2$, $\forall u \in U, \exists v \in U.$

$$< g(y_1,u), \frac{\partial}{\partial y_1}\Delta(y_1,y_2) > + < g(y_2,v), \frac{\partial}{\partial y_2}\Delta(y_1,y_2) > \leq 0$$

Then $\Gamma(y_0)$ ***has a*** *limit value* $V_t(y_0) \xrightarrow[t\to+\infty]{} V^*(y_0)$. ***The convergence of*** $(V_t)_t$ ***to*** V^* ***is uniform over*** $G(y_0)$***, and the value of*** $\Gamma(y_0)$ ***is uniform.***

• This result can be applied to example 4, with $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$ (L^1-norm). In this example, we have for each y_1, y_2 and u: $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$ as soon as $t \geq 0$ is small enough.

Example 4: in IR^2. The initial state is $y_0 = (0, 0)$ and $U = [0, 1]$, and the cost is $h(y) = 1 - y_1(1 - y_2)$.

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$

Further Generalizations

Theorem 9 ***(H1)*** h ***is uniformly continuous in*** y ***on*** $\bar{Z}$ ***uniformly in*** u***. And for each*** y ***in*** $\bar{Z}$***, either*** h ***does not depend on*** u ***or the set*** $\{(g(y,u), h(y,u)) \in I\!R^d \times [0,1],\ u \in U\}$ ***is closed.***
(H2): $\exists \Delta : I\!R^d \times I\!R^d \longrightarrow I\!R_+$***, vanishing on the diagonal*** $(\Delta(y,y) = 0)$ ***and symmetric*** $(\Delta(y_1,y_2) = \Delta(y_2,y_1)$ ***), and a unniformly continuous function*** $\hat{\alpha} : I\!R_+ \longrightarrow I\!R_+$ ***s.t.*** $\hat{\alpha}(t) \xrightarrow[t \to 0]{} 0$ ***satisfying:***

$a)$ $\forall$ ***sequence*** $(z_n)_n \subset Z$***,*** $\forall \varepsilon > 0$***,*** $\exists n$***,*** $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$***.***

$b)$ $\forall (y_1,y_2) \in \bar{Z}^2$, $\forall u \in U$***,*** $\exists v \in U$ ***such that***

$$D \uparrow \Delta(y_1,y_2)(g(y_1,u), g(y_2,v)) \leq 0,\ h(y_2,v) - h(y_1,u) \leq \hat{\alpha}(\Delta(y_1,y_2)).$$

Then $\Gamma(y_0)$ ***has a uniform value*** $\lim_{t\to\infty} V_t = V^*$.

Remarks

- **Although Δ may not satisfy the triangular inequality nor the separation property, it may be seen as a "distance" adapted to the problem $\Gamma(y_0)$.**
- $D\uparrow$ **is the contingent epi-derivative (which reduces to the upper Dini derivative if Δ is Lipschitz)** $D{\uparrow}\Delta(z)(\alpha) = \liminf_{t\to 0^+,\alpha'\to\alpha} \frac{1}{t}(\Delta(z+t\alpha') - \Delta(z))$**. If Δ is differentiable, the condition** $D\uparrow\Delta(y_1,y_2)(g(y_1,u),g(y_2,v)) \leq 0$ **just reads:** $< g(y_1,u), \frac{\partial}{\partial y_1}\Delta(y_1,y_2) > + < g(y_2,v), \frac{\partial}{\partial y_2}\Delta(y_1,y_2) > \leq 0$**.**

• The assumption: "$\{(g(y,u),h(y,u)) \in IR^d \times [0,1],\ \ u \in U\}$ **closed" could be checked for instance if** U **is compact and if** h **and** g **are continuous with respect to** (y,u)**.**

• $H2a)$ **is a precompacity condition. It is satisfied as soon as** $G(y_0)$ **is bounded. cf Renault 2008**

• Notice that $H2$ **is satisfied with** $\Delta = 0$ **if we are in the trivial case where** $\inf_u h(y,u)$ **is constant.**

On Uniform Value

Definition 10 $\Gamma(y_0)$ ***has a*** *uniform value* ***if*** $\exists V(y_0)$ ***and if:***

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^{t} h(y(s,u,y_0), u(s))ds \leq V(y_0) + \varepsilon.$$

- **Example 5: in** IR^2**,** $y_0 = (0,0)$**, control set** $U = [0,1]$**,** $y'(t) = (y_2(t), u(t))$**, and** $h(y_1,y_2) = 0$ **if** $y_1 \in [1,2]$**,** $= 1$ **otherwise.**

We have $u(s) = y_2'(s) = y_1''(s)$**,**

Interpretation: u **"acceleration",** y_2 **"speed",** y_1 **the "position".**

If $u = \varepsilon$ **constant, then** $y_2(t) = \sqrt{2\varepsilon y_1(t)}$ $\forall t \geq 0$**.**

Limit Value: $V_T(y_0) \xrightarrow[T \to \infty]{} 1/2$

No Uniform Value.

Optimal control with discounted facteur $\lambda \to 0^+$

We define $\Theta_\lambda(y_0) := \inf\limits_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s,u,y_0), u(s)) ds,$

Theorem 11 ***(Oliu-Barton Vigeral 2010) the following uniform limit in*** K ***exists*** $\lim_{\lambda \to 0^+} \Theta_\lambda(y_0)$

iff

the following uniform limit in K ***exists*** $\lim_{t \to \infty} V_t(y_0)$

Question Application to different concepts of means

Open Problems

Differential Game at horizon t:

$$V_t(y_0) := \text{"} \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \text{"} \frac{1}{t} \int_{s=0}^{t} h(y(s,u,v,y_0), u(s), v(s)) ds,$$

where $s \mapsto y(s,u,y_0)$ denotes the solution to

$$y'(s) = g(y(s), u(s), v(s))), \quad y(0) = y_0.$$

OPEN PROBLEM : Existence of a limit of $V_t(y_0)$ as $t \to \infty$.

Only Partial results:

- **When the Hamiltonian is coercive (hence ergodicity and the limit is y independent)Alvarez-Bardi ...**
- **For nonconvex and non coercive Hamiltonian in IR^2 Cardaliagu**

Averaging Problem for singularly perturbed system

$$\begin{cases} i) \quad x'(s) = f\big(x(s), y(s), u(s)\big), \; x(0) = x, \; s \in [0, T] \\ ii) \; \varepsilon y'(s) = g\big(x(s), y(s), u(s)\big) \; y(0) = y, \end{cases} \tag{1}$$

Change of variable $\tau = \frac{t}{\varepsilon}$**,** $(X(\tau), Y(\tau), U(\tau)) = (x(\varepsilon\tau), y(\varepsilon\tau), u(\varepsilon\tau))$

$$\begin{cases} X'(\tau) = \varepsilon f\big(X(\tau), Y(\tau), U(\tau)\big), \; X(0) = x, \tau \in [0, \frac{T}{\varepsilon}] \\ Y'(\tau) = g\big(X(\tau), Y(\tau), U(\tau)\big), \; Y(0) = y. \end{cases} \tag{2}$$

Take $\varepsilon = 0$ **in (2).We have the following associated system:**

$$y'(\tau) = g\big(x, y(\tau), u(\tau)\big), \; y(0) = y, \tag{3}$$

$y_x(\cdot, u, y)$ **denotes the unique solution of (3).**

Averaging method

We suppose that f and g are Lipschitz and there is a compact set $M \times N$ which is is invariant by (1) for all ε.

$A(x,y,S,u) = \frac{1}{S}\int_0^S f(x, y_x(\tau,u,y), u(\tau))d\tau,$

$F(x,y,S) = \{A(x,y,S,u); u \in \mathcal{U}\}$

Theorem 12 ***Gaitsgory, Grammel*** ***If*** $\exists \gamma : I\!R \to I\!R_+$ ***with*** $\lim_{S\to+\infty}\gamma(S) = 0$ ***and a Lipschitz set-valued map*** $\bar{F} : M \to I\!R^b b$ ***with compact convex nonempty values such that***

$$d(co\, cl F(x,y,S), \overline{F}(x)) \leq \gamma(S),\ \forall (x,y) \in M \times N, \forall S > 0,$$

then $\forall x, y$ ***the solutions of the differential inclusion***

$$x'(s) \in \overline{F}\big(x(s)\big),\ x(0) = x. \tag{4}$$

approximate the solutions of the singularly perturbed system (1) in the following sense:

For any $\varepsilon > 0$**, and any** $T > 0$ **there exists** $M(T,\varepsilon) > 0$ **with** $\lim_{\varepsilon\to 0} M(T,\varepsilon) = 0$ **such that**

a) For any family of solutions $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ **to (1) there exists a solution** $x(\cdot)$ **to (4) such that**

$$\sup_{t\in[0,T]} \|x_\varepsilon(t) - x(t)\| \leq M(T,\varepsilon).$$

b) Conversely fix $x(\cdot)$ **a solution to (4) then for any** ε **small enough there exists a solution** $\big(x_\varepsilon(\cdot), y_\varepsilon(\cdot)\big)$ **to (1) such that**

$$\sup_{t\in[0,T]} \|x_\varepsilon(t) - x(t)\| \leq M(T,\varepsilon).$$

cf also Wattbled, M.Q Wattbled ...

QUESTION ases and conditions where $\overline{F}$ **may depend on** y**.**

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Thank You for your Attention